Global existence for some configurations of nearly parallel vortex filaments

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A model for nearly parallel vortex filaments

In a 3D homogeneous incompressible fluid a vortex filament is a a vortex tube with infinitesimal cross section: the vorticity is a singular measure supported along a curve in $\mathbb{R}^3$.

Klein-Majda-Damodaran 95: for N vortex filaments nearly parallel to e_3 parametrized by

$$(x_j(t,\sigma), y_j(t,\sigma), \sigma),$$

of circulation Γ_j, the evolution of $\Psi_j(t,\sigma) = x_j(t,\sigma) + iy_j(t,\sigma)$ is modeled by the 1-D Schrödinger system

$$\begin{cases} i\partial_t \Psi_j + \Gamma_j \partial_\sigma^2 \Psi_j + \sum_{k \neq j} \Gamma_k \frac{\Psi_j - \Psi_k}{|\Psi_j - \Psi_k|^2} = 0, \ 1 \leq j \leq N \, . \end{cases}$$

In the case of exact parallel filaments, $\Psi_j(t,\sigma) = X_j(t)$, we get the evolution of point vortex system

$$\begin{cases} i\partial_t X_j + \sum_{k \neq j} \Gamma_k \frac{X_j - X_k}{|X_j - X_k|^2} = 0, \ 1 \leq j \leq N \, . \end{cases}$$

Some results on the point vortex system dynamics

- $\Gamma_j > 0$ global existence using conservation laws,
- N=2, global existence since $|X_1(t) - X_2(t)|$ is conserved, $(X_1(t), X_2(t))$ rotate or translate,
- N=3 explicit collapse for certain configurations: shrinking turning triangle, Aref 79,
- N=3 vortex points placed at the vertices of an equilateral triangle rotate or translate,
- N=3 vortex placed at the ends and the middle of a segment, $\Gamma_j = \Gamma$, rotate or translate,
- vortex points placed at the $N \geq 4$ vertices of a regular polygon, $\Gamma_j = \Gamma$, rotate,
- also the vertices of regular polygons, with $\Gamma_j = \Gamma$, together with the center of the polygon form a relative equilibrium configuration,
- Kelvin's conjecture 1878: the polygon configuration is stable iff $N \leq 7$, Novikov 75, Kurakin-Yudovich 02.

Results on the nearly parallel vortex filaments

On perturbations of exact parallel filaments, $\Psi_j(t,\sigma) = X_j(t) + u_j(t,\sigma)$:

- Klein-Majda-Damodaran 95:
  $N = 2$, the linearized system is stable if $\Gamma_1/\Gamma_2 > 0$ and unstable if $\Gamma_1/\Gamma_2 < 0$. Numerical computations on the perturbations suggest global existence in the first case and collision in the second.
- Kenig-Ponce-Vega 03:
  $\forall N$ local existence for any $(X_j(0))$ and small H^1 perturbations $(u_j(0))$, existence time $\gtrsim |\log(\Sigma \|u_j(0)\|_{H^1})|$.

  $N = 2$ global existence for any $(X_j(0))$, $\Gamma_j = \Gamma > 0$.

  $N = 3$ global existence for $(X_j(0))$ equilateral triangle, $\Gamma_j = \Gamma > 0$.

  The global existence proofs are based on

  $$|X_j(t) - X_k(t)| = d, \forall\, 1 \leq j \neq k \leq N$$

  which insures the conservation of the energy $\mathcal{E}(t)$
  $\Sigma \int |\partial_\sigma \Psi_j(t,\sigma)|^2 \, d\sigma + \Sigma \int - \ln\left(\frac{|\Psi_{jk}(t,\sigma)|^2}{|X_{jk}(t)|^2}\right) + \left(\frac{|\Psi_{jk}(t,\sigma)|^2}{|X_{jk}(t)|^2} - 1\right) d\sigma.$
  The solutions satisfy $\frac{3}{4} \leq \frac{|\Psi_j(t,\sigma) - \Psi_k(t,\sigma)|}{|X_j(t) - X_k(t)|} \leq \frac{5}{4}$.

Results on the nearly parallel vortex filaments

Theorem (B-M 11)

$N=4$ global existence for $(X_j(0))$ vertices of a square centered at 0, $\Gamma_j=\Gamma>0$ and $(\Psi_1+\Psi_3)(0,\sigma)=(\Psi_2+\Psi_4)(0,\sigma)=0$ $\forall\sigma$.

$N=4$ local existence for $(X_j(0))$ vertices of a square, $\Gamma_j=\Gamma>0$, existence time $\gtrsim \min\{\mathcal{E}(0)^{-\frac{1}{4}}\Sigma\|u_j(0)\|_{L^2}^{-\frac{1}{2}}, \mathcal{E}(0)^{-\frac{1}{3}}\}$ with $\mathcal{E}(0)\lesssim \Sigma\|u_j(0)\|_{H^1}^2$.
The solutions satisfy $\frac{3}{4}\leq \frac{|\Psi_j(t,\sigma)-\Psi_k(t,\sigma)|}{|X_j(t)-X_k(t)|}\leq \frac{5}{4}$.

- the rhombus shape are conserved since $(-\Psi_3,-\Psi_4,-\Psi_1,-\Psi_2)$ is also solution,
- for global $\exists$ the inertia centrum satisfies $\sum\Psi_j(t,\sigma)=\sum X_j(t)=0$,
- $\forall T$ there are perturbations on $[0,T]$, with $\mathcal{E}(0)\ll 1\sim \Sigma\|u_j(0)\|_{H^1}^2$,
- $|X_j(t)-X_k(t)|$ conserved, but not the same.

Results on the nearly parallel vortex filaments

Let $(X_j(t))$ be the vertices of a rotating regular polygon of radius 1 (with or without its center). We consider dilation-rotation type perturbations that preserve the polygonal shape $\forall t, \sigma$,

$$\Psi_j(t,\sigma) = X_j(t)\Phi(t,\sigma).$$

Theorem 2 (B-M 11)

- If $\Phi(0) - 1$ is small in H^1 then we have global existence and $\frac{3}{4} \leq \frac{|\Psi_j(t,\sigma)-\Psi_k(t,\sigma)|}{|X_j(t)-X_k(t)|} = |\Phi(t,\sigma)| \leq \frac{5}{4}$, $\Psi_j(t,\sigma) \overset{|\sigma|\to\infty}{\longrightarrow} X_j(t)$.
- If $\mathcal{E}(0) = \frac{1}{2}\int |\partial_\sigma \Phi(0)|^2 + \frac{\omega}{2}\int \left(|\Phi(0)|^2 - 1 - \ln|\Phi(0)|^2\right)$ is small then we have global existence and $\frac{3}{4} \leq \frac{|\Psi_j(t,\sigma)-\Psi_k(t,\sigma)|}{|X_j(t)-X_k(t)|} \leq \frac{5}{4}$. Moreover, if $\Phi(0,\sigma) \overset{|\sigma|\to\infty}{\longrightarrow} 1$ then $\Psi_j(t,\sigma) \overset{|\sigma|\to\infty}{\longrightarrow} X_j(t)$.

Results on the nearly parallel vortex filaments

- Gross-Pitaevskii type dynamics for the perturbation Φ

$$i\partial_t\Phi + \partial_\sigma^2\Phi + \omega\frac{\Phi}{|\Phi|^2}(1-|\Phi|^2) = 0,$$

  with $\omega \in \mathbb{R}^{+*}$ the rotating speed of the point vortices,
- conservation of the energy

$$\mathcal{E}(t) = \frac{1}{2}\int|\partial_\sigma\Phi(t)|^2 + \frac{\omega}{2}\int\left(|\Phi(t)|^2 - 1 - \ln|\Phi(t)|^2\right).$$

- the energy space contains small rotation type perturbations and grey solitons (finite energy travelling waves of G-P),
- existence of travelling waves,
- in progress: collisions,
- for shift type perturbations $\Psi_j(t,\sigma) = X_j(t) + u(t,\sigma)$, linear Schrödinger dynamics.

Proof of Theorem 2

Lemma 1

Energy $\mathcal{E}(t)$ small enough implies $\||\Phi(t)|^2 - 1\|_{L^\infty} \leq \frac{1}{4}$.

The function $f(x) = x - 1 - \log x$ is positive and convexe, and vanishes only at $x = 1$. If $\exists \sigma_0$ such that $|\Phi(t, \sigma_0)| > \sqrt{\frac{5}{4}}$ then

$|\Phi(t,\sigma)| \geq |\Phi(t,\sigma_0)| + |\int_{\sigma_0}^{\sigma} \partial_x \Phi(t,x) dx| \geq \sqrt{\frac{5}{4}} - \sqrt{2\mathcal{E}(\Phi(t))|\sigma - \sigma_0|}$,

and $|\Phi(t,\sigma)| > \sqrt{\frac{9}{8}}$ sur $I = [\sigma_0 - \frac{1}{500\,\mathcal{E}(t)}, \sigma_0 + \frac{1}{500\,\mathcal{E}(t)}]$. Finally,

$$\mathcal{E}(t) \geq \frac{1}{2} f\left(\frac{9}{8}\right)|I| = \frac{1}{1000\,\mathcal{E}(t)} f\left(\frac{9}{8}\right),$$

contradiction for $\mathcal{E}(t)$ small enough.
Since $\frac{1}{2}(x-1)^2 \leq x - 1 - \ln x \leq 10(x-1)^2$ on $[\frac{3}{4}, \frac{5}{4}]$ we have:

Lemma 2

$\||\Phi(t)|^2 - 1\|_{L^\infty} \leq \frac{1}{4}$ implies the comparaison of the energies:
$\mathcal{E}_{GP}(t) = \frac{1}{2}\|\partial_\sigma \Phi(t)\|_{L^2}^2 + \frac{\omega}{4}\||\Phi(t)|^2 - 1\|_{L^2}^2 \leq \mathcal{E}(t) \leq 5\,\mathcal{E}_{GP}(t)$.

Proof of Theorem 2: resolution in $1 + H^1$

Similar arguments for Gross-Pitaevskii in $1 + H^1$ (Béthuel-Saut 99, B-Vega 08) : We first solve locally the Schrödinger-type equation satisfied by $u(t) = \Phi(t) - 1$.

Since $\Phi(0) - 1$ is small in H^1, Lemma 2 and Gagliardo-Niremberg imply $\mathcal{E} = \mathcal{E}(0)$ small. Then, by Lemma 1, the quotient $\frac{1}{|\Phi(t)|^2}$ will remain uniformly bounded.

The existence time will then depend on the H^1 norm of $u(t)$. The $\dot{H}^1$ norm stays bounded in time by the energy, and the L^2 norm satisfies

$$\partial_t \int |u(t)|^2 = \Im\omega \int \frac{1 + u(t)}{|1 + u(t)|^2}(1 - |1 + u(t)|^2)\overline{u}(t)$$

$$= \Im\omega \int \frac{(1 - |1 + u(t)|^2)\overline{u}(t)}{|1 + u(t)|^2} \leq |\omega| \| 1 - |\Phi(t)|^2 \|_{L^2} \|u(t)\|_{L^2} \leq |\omega| 2\sqrt{\mathcal{E}} \|u(t)\|_{L^2},$$

so $\|u(t)\|_{L^2} \lesssim t$. By re-iterating the local in time argument we get the global existence.

Proof of Theorem 2: resolution in the energy space

Similar arguments for Gross-Pitaevskii in the energy space (Zhidkov 87, Gérard 06) : We solve locally in time by a fixed point argument for the operator

$A(w)(t) = \omega \int_0^t e^{i(t-\tau)\partial_\tau^2} \frac{e^{i\tau\partial_\sigma^2}\Phi(0)+w(\tau)}{|e^{i\tau\partial_\sigma^2}\Phi(0)+w(\tau)|^2} \left(1 - |e^{i\tau\partial_\sigma^2}\Phi(0) + w(\tau)|^2\right)$, on $\sup_{0\le t\le T} \|w(t)\|_{H^1} \le \epsilon$.

By Lemma 1, $|\Phi(0)| \ge \frac{\sqrt{3}}{2}$.

On the other hand, since the symbol of $e^{it\partial_\sigma^2} - 1$ is $\frac{e^{-it\xi^2}-1}{\xi}\xi$,

$\|e^{i\tau\partial_\sigma^2}\Phi(0) - \Phi(0)\|_{H^1} \le C(1+\tau^{\frac{1}{2}})\|\partial_\sigma\Phi(0)\|_{L^2} \le C(1+\tau^{\frac{1}{2}})\sqrt{\mathcal{E}}$.

By taking ϵ, T small with respect to $\mathcal{E}$, $\frac{1}{|e^{i\tau\partial_\sigma^2}\Phi(0)+w(\tau)|^2}$ will stay uniformly bounded.

We obtain $\|A(w)(t)\|_{H^1} \le C(\epsilon)t(C+\sqrt{\mathcal{E}})$, and we deduce the existence of a local solution for ϵ, T small with respect to $\mathcal{E}$.

By re-iterating the local in time argument we get the global existence.

Proof of Theorem 1: local existence K-P-V

For a a perturbation $u_{j,0}(\sigma) = \Psi_j(0,\sigma) - X_j(0)$ small in H^1 $\exists T^* \in]0,\infty]$ maximal time such that on $[0, T^*[\times\mathbb{R}$

$$\frac{3}{4}|X_j(t) - X_k(t)| < |\Psi_j(t,\sigma) - \Psi_k(t,\sigma)| < \frac{5}{4}|X_j(t) - X_k(t)|,$$

so for $T \leq T^*$ the fixed point operator can be bounded by

$$\Sigma\|A(u_j)\|_{L^\infty([0,T],H^1)} \leq \Sigma\|u_{j,0}\|_{H^1} + C(|X_{kl}|)T\,\Sigma\|u_j\|_{L^\infty([0,T],H^1)}.$$

For T small enough we obtain on $[0, T]$ a solution (u_j) such that

$$\Sigma\|u_j\|_{L^\infty([0,T],H^1)} \leq 2\Sigma\|u_j(0)\|_{H^1}.$$

The solution can be extended -although the H^1 norm might grow- on $[0, T^*]$ with $|\log(\Sigma\|u_j(0)\|_{H^1})| \lesssim T^*$. For showing the global existence it is enough to get, if T^* is supposed finite, the contradiction

$$\frac{3}{4}|X_j(T^*) - X_k(T^*)| < |\Psi_j(T^*,\sigma) - \Psi_k(T^*,\sigma)| < \frac{5}{4}|X_j(T^*) - X_k(T^*)|.$$

Proof of Theorem 1: towards global existence K-P-V

The following quantities are conserved

$$\mathcal{H} = \sum_j \int |\partial_\sigma \Psi_j(t,\sigma)|^2 \, d\sigma - \sum_{j\neq k} \int \ln\left(\frac{|\Psi_{jk}(t,\sigma)|^2}{|X_{jk}|^2(t)}\right) d\sigma,$$

$$\mathcal{A} = \sum_j \int \left(|\Psi_j(t,\sigma)|^2 - |X_j(t)|^2\right) d\sigma,$$

$$\mathcal{T} = \sum_{j\neq k} \int \left(|\Psi_{jk}(t,\sigma)|^2 - |X_{jk}(t)|^2\right) d\sigma.$$

Let

$$\mathcal{I}(t) = \sum_{j\neq k} \int \left(\frac{|\Psi_{jk}(t)|^2}{|X_{jk}(t)|^2} - 1\right) d\sigma.$$

Since $-\ln(x) + (x-1) \geq \frac{1}{2}(x-1)^2$ for $x \in \left[\frac{3}{4}, \frac{5}{4}\right]$, on $[0, T^*]$ we have

$$\mathcal{E}(t) = \mathcal{H} + \mathcal{I}(t) \geq \frac{1}{2} \sum_{j\neq k} \left\| \frac{|\Psi_{jk}(t)|^2}{|X_{jk}(t)|^2} - 1 \right\|_{L^2}^2 + \sum_j \|\partial_\sigma \Psi_j(t)\|_{L^2}^2.$$

Proof of Theorem 1: towards global existence K-P-V

By Gagliardo-Niremberg, on $[0, T^*]$,

$$\left\| \frac{|\Psi_{jk}(t,\sigma)|^2}{|X_{jk}(t)|^2} - 1 \right\|_{L^\infty} \leq C\mathcal{E}(t)^{\frac{1}{2}} \frac{\|\Psi_{jk}(t)\|_{L^\infty}^{\frac{1}{2}} \mathcal{E}(t)^{\frac{1}{2}}}{|X_{jk}(t)|} \leq C\mathcal{E}(t),$$

so if $\mathcal{E}(t)$ stays small enough on $[0, T^*]$ then $|\Psi_{jk}(t,\sigma)|$ is close enough to $|X_{jk}(t)|$ such that

$$\frac{3}{4}|X_j(T^*) - X_k(T^*)| < |\Psi_j(T^*,\sigma) - \Psi_k(T^*,\sigma)| < \frac{5}{4}|X_j(T^*) - X_k(T^*)|,$$

which is the contradiction that implies the global existence.

In the (K-P-V) cases, $|X_{jk}(t)| = d$ so $\mathcal{E}(t) = \frac{\mathcal{T}}{d}$ is conserved, and global existence is obtained for small $\mathcal{E}(0)$.

Actually, in the cases of Theorem 2, $\mathcal{E}(t) = \mathcal{E}(\Phi(t))$ is conserved and the global existence in $1 + H^1$ can be obtained also this way.

Proof of Theorem 1

Control of $\mathcal{E}(t)$:

$$\mathcal{E}(t) = -\mathcal{H} + \frac{1}{2}\mathcal{T} - \mathcal{A} + \frac{1}{2}(\|u_1(t) + u_3(t)\|_{L^2}^2 + \|u_2(t) + u_4(t)\|_{L^2}^2).$$

$\rightsquigarrow$ $\mathcal{E}(t)$ conserved and implies global existence for rhombus type perturbations,

$\rightsquigarrow$ for general perturbations

$$|\mathcal{E}(t)| \leq |\mathcal{E}(0)|$$

$$+t^2 \sup_{\tau\in[0,t]} |\mathcal{E}(\tau)|^{\frac{3}{2}}(\Sigma\|u_j(0)\|_{L^2} + t \sup_{\tau\in[0,t]} |\mathcal{E}(\tau)|^{\frac{1}{2}}),$$

so $T^* \gtrsim \min\left\{\frac{1}{\sqrt{\mathcal{E}(\Phi(0))^{\frac{1}{2}}\Sigma\|u_j(0)\|_{L^2}}}, \frac{1}{\mathcal{E}(0)^{\frac{1}{3}}}\right\}$.